A02

Thursday, January 6, 2022 6:08 PM

name: Hans Oberschelp
email: hoberschelp@math.ucdavis.edu
office: MSB 2202

1) Roll two dice: $\mathbb{P}(\text{sum of the dice is} \leq 6) = \frac{\#\text{ "good" outcomes}}{\#\text{ total outcomes}}$

Die #1

Die #2	1	2	3	4	5	6
1	✓	✓	✓	✓	✓	✗
2	✓	✓	✓	✓	✗	
3	✓	✓	✓	✗		
4	✓	✓	✗			
5	✓	✗				
6	✗					

$= \frac{15}{36} = \frac{5}{12}$

What is wrong with this approach:

possible sums: (2)(3)(4)(5)(6) 7, 8, 9, 10, 11, 12 (the first 11)

$\neq \frac{5}{11}$

$\mathbb{P}(\text{sum} \leq 6) + \mathbb{P}(\text{sum} = 7) + \mathbb{P}(\text{sum} \geq 8) = 1$

$= x \qquad \frac{1}{6} \qquad = x$

$2x + \frac{1}{6} = 1$

$\Rightarrow 2x = \frac{5}{6}$

$\Rightarrow x = \frac{5}{12}$

2) Flip a coin 3 times. $\frac{1}{2^3}$

- 2 choices for H/T in each flip
- 3 flips

$\mathbb{P}(\text{"HHH"}) = \frac{1}{8}$

$\mathbb{P}(\text{"HTH"}) = \frac{1}{8}$

... somewhere in sequence) $= \frac{3}{16}$

P("HTH") = 8

Flip a coin 4 times. P("HHH" appears anywhere in sequence) = $\frac{3}{16}$

HHHH HHHT HHTH HHTT
HTHH HTHT HTTH HTTT
THHH THHT THTH THTT
TTHH TTHT TTTH TTTT

P("HTH" appears) = $\frac{4}{16}$

3) In 1991 there was a lottery called Lotto America.
Choose 6 distinct numbers between 1 and 54.

$$P(\text{win jackpot}) = \frac{\#\text{ good outcomes}}{\#\text{ total outcomes}} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{54\cdot53\cdot52\cdot51\cdot50\cdot49} = \frac{1}{26 \text{ million}}$$

In 1992, Lotto America changed its name to Powerball, and also changed the rules.
Choose 5 distinct white numbers between 1 and 45.
Choose 1 red number between 1 and 45.

$$P(\text{win jackpot}) = \frac{\#\text{ good outcomes}}{\#\text{ total outcomes}} = \frac{\overbrace{5\cdot4\cdot3\cdot2\cdot1}^{\text{winning white balls}}\cdot\overbrace{1}^{\text{winning red ball}}}{45\cdot44\cdot43\cdot42\cdot41\cdot45} = \frac{1}{55 \text{ million}}$$

A01

Thursday, January 20, 2022 5:05 PM

Coupon Collector

Suppose each week you are awarded a random coupon #1, #2, --, or #10.
If you collect coupons for 20 weeks, what is the probability that you complete a set of 10?

Let A_i be the event that coupon #i is still missing after 20 weeks.

$\mathbb{P}(\text{fail to finish our set}) = \mathbb{P}(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } __ \text{ or } A_{10})$
$= \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup __ \cup A_{10})$

$\mathbb{P}(\text{complete set}) = 1 - \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup __ \cup A_{10})$
$= 1 - \mathbb{P}(A_1) - \mathbb{P}(A_2) - __ - \mathbb{P}(A_{10}) + \mathbb{P}(A_1 \cap A_2) + __ + \mathbb{P}(A_9 \cap A_{10}) - \mathbb{P}(A_1 \cap A_2 \cap A_3) + __$

$\mathbb{P}(A_1) = \frac{9^{20}}{10^{20}} = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \cdots$

$\mathbb{P}(A_1 \cap A_2 \cap __ \cap A_{10})$

$\mathbb{P}(A_1 \cap A_2) = \frac{8^{20}}{10^{20}} = \mathbb{P}(A_3 \cap A_5) = __$

$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{7^{20}}{10^{20}}$

$$\mathbb{P}(\text{complete set}) = 1 - 10 \cdot \frac{9^{20}}{10^{20}} + \binom{10}{2} \cdot \frac{8^{20}}{10^{20}} - \binom{10}{3} \cdot \frac{7^{20}}{10^{20}} + \cdots + \binom{10}{10} \cdot \frac{0^{20}}{10^{20}}$$

$10 = \binom{10}{1}$

$1 = \binom{10}{0} \cdot \frac{10^{20}}{10^{20}}$

$$= \sum_{n=0}^{10} (-1)^n \cdot \binom{10}{n} \cdot \frac{(10-n)^{20}}{10^{20}} \approx 0.21$$

Lets say there are 100 coupons in a set, and we wait 200 weeks.

$\mathbb{P}(\text{complete set}) \approx 2 \times 10^{-8}$

A01 Quiz

Thursday, January 27, 2022 5:46 PM

It's Holloween and the bowl of candy in front of you has:

- 15 chocolate bars
- 5 lollipops
- 10 packs of gummy worms
- 1 licorice stick

You grab 4 pieces of candy at random to put in your basket.

(a) What is the probability that all the candy you grab is of the same type?

(b) What is the probability that you grab 1 of each type of candy.

A02 Quiz

Thursday, January 27, 2022 6:47 PM

You are a reporter at the 2022 Beijing Winter Olympics, and you are pulling aside athletes for interviews. In front of you there are

- 10 skiiers
- 10 ice skaters
- 10 curlers

You grab 5 athletes at random and take them into the press room.

(a) What is the probability that there is at least one sport (out of skiing, skating and curling) with no athletes in the press room.

(b) What is the probability that you have at least one complete set of "three oldest athletes of sport n" for at least one of the sports.

Friday, March 11, 2022 4:04 PM

In probability, we do experiment with random outcomes, and we ask what is the probability that a certain event occurs.
(Events are collections of outcomes)

ex. I flip a coin 3 times. The outcomes are:
HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

"flipping exactly one head" is an event. It is the collection of the outcomes: HTT, THT, TTH

$\mathbb{P}(\text{flip exactly one head}) = \frac{3}{8}$

Def: The complement of an event is the collection of outcomes not in that event.
We denote the complement of A (an event) as A^c.

ex. $\mathbb{P}((\text{flip exactly one head})^c) = \mathbb{P}(\text{flip 0, 2 or 3 heads}) = \frac{5}{8}$

Def: Borrowing from set theory, we use $A \cup B$ to represent the even "A or B" (and or is inclusive)
and we use $A \cap B$ to represent "A and B".

ex. $\mathbb{P}((\text{flip exactly one heads}) \cup (\text{first flip is tails})) = \frac{5}{8}$
$\mathbb{P}((\text{last flip is heads}) \cap (\text{first flip is tails})) = \frac{2}{8}$

Here are some important theorems about probability. Assume A and B are events.

- $0 \le \mathbb{P}(A) \le 1$
- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- $\mathbb{P}(A \cup B) \ge \mathbb{P}(A)$ and $\mathbb{P}(A \cup B) \ge \mathbb{P}(B)$
- $\mathbb{P}(A \cap B) \le \mathbb{P}(A)$ and $\mathbb{P}(A \cap B) \le \mathbb{P}(B)$
- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$
- If A and B are disjoint (i.e. $\mathbb{P}(A \cap B) = 0$ i.e. "A and B" is impossible)
  then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$
- $\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$ no matter what

Thm: Inclusion-Exclusion Formula

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$
$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$
$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} \mathbb{P}(A_1 \cap \cdots \cap A_n)$$

Def: The conditional probability of A given B is defined

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{★} \quad \mathbb{P}(A|B) \cdot \mathbb{P}(B) = \mathbb{P}(A \cap B)$$

ex. Bag W has 3 white balls and 5 red balls.
Bag R has 2 white balls and 2 red balls.
Pick a bag at random and take a ball from it. Don't replace the ball.
If you drew a red ball, take another ball from bag R.
If you drew a white ball, take another ball from bag W.

$$\begin{aligned}
P(\text{2 red balls}) &= P((\text{2 red balls and the first one is from R}) \cup (\text{2 red balls and the first is from W})) \\
&= P((\text{2 red balls}) \cap (\text{first red from R})) + P((\text{2 red balls}) \cap (\text{first red from W})) \\
&= P(\text{first red from R}) \cdot P(\text{2 red balls} \mid \text{first red from R}) + P(\text{first red from W}) \cdot P(\text{2 red balls} \mid \text{first red from W}) \\
&= \left(\tfrac{1}{2} \cdot \tfrac{3}{4}\right) \cdot \tfrac{1}{3} + \left(\tfrac{1}{2} \cdot \tfrac{5}{8}\right) \cdot \tfrac{2}{4}
\end{aligned}$$

Thm: Bayes Formula

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)}$$

Or, more generally, if $P(F_1 \cup \cdots \cup F_n) = 1$ and $F_1, \ldots, F_n$ are disjoint then

$$P(F_i|B) = \frac{P(B|F_i) \cdot P(F_i)}{P(B|F_1) \cdot P(F_1) + \cdots + P(B|F_n) \cdot P(F_n)}$$

ex. When there is a wildfire nearby, the sky is always smoky.
But even when there isn't a nearby wildfire, the skies are smoky $\frac{1}{50}$ of the time.
If there is usually a $\frac{1}{100}$ chance that a wildfire is nearby,
what is the probability that a wildfire is nearby if you see that the sky is smoky.

$$\begin{aligned}
P(\underset{A}{\text{wildfire}} \mid \underset{B}{\text{smoke}}) &= \frac{P(\text{smoke} \mid \text{wildfire}) \cdot P(\text{wildfire})}{P(\text{smoke} \mid \text{wildfire}) \cdot P(\text{wildfire}) + P(\text{smoke} \mid \text{no wildfire}) \cdot P(\text{no wildfire})} \\
&= \frac{1 \cdot \frac{1}{100}}{1 \cdot \frac{1}{100} + \frac{1}{50} \cdot \frac{99}{100}} = \frac{1}{100\left(\frac{1}{100} + \frac{1}{50} \cdot \frac{99}{100}\right)} = \frac{1}{1 + \frac{99}{50}} = \frac{1}{\frac{149}{50}} = \frac{50}{149} \approx \frac{1}{3}
\end{aligned}$$

Def: Events A and B are <u>independent</u> if $P(A \cap B) = P(A) \cdot P(B)$

ex. Let X be a uniform([0,1]) random variable.
Let $A = \{X \in [0, \frac{1}{2}]\}$
Let $B = \{X \in [\frac{1}{4}, \frac{3}{4}]\}$

$$P(A \cap B) = P(X \in [\tfrac{1}{4}, \tfrac{1}{2}]) = \frac{\frac{1}{2} - \frac{1}{4}}{1 - 0} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$$

Therefor A and B are independent.

$P(A) \cdot P(B) \iff P(A|B) = P(A) \iff P(B|A) = P(B)$

Therefor A and D are independent.

Thm: $P(A \cap B) = P(A) \cdot P(B) \iff P(A|B) = P(A) \iff P(B|A) = P(B)$

Def: A discrete random variable X has finitely or countably many outcomes, and all its outcomes are assigned a real number.

ex. Roll two 6-sided dice and let X be the bigger of the two numbers that you roll.

$P(X=2) = \frac{3}{36} = \frac{1}{12}$

(11) 12 21 22 (13) (23) (33)

Def: The function $p(k) = P(X=k)$ is called the probability mass function of X.

i.e. $p(2) = \frac{1}{12}$

ex. Flip 3 coins and let Y = #Heads − #Tails
What is the PMF of Y?

$P(Y=-3) = \frac{1}{8}$ $P(Y=-1) = \frac{3}{8}$ $P(Y=1) = \frac{3}{8}$ $P(Y=3) = \frac{1}{8}$

Note: We can define r.v.'s as functions of other r.v.'s

ex. Flip 3 coins and let Y = #Heads − #Tails
Let $Z = Y^4$. What is the PMF of Z?

$Y=-3 \Rightarrow Z=81$
$Y=-1 \Rightarrow Z=1$
$Y=1 \Rightarrow Z=1$
$Y=3 \Rightarrow Z=81$

$P(Z=81) = P((Y=-3) \cup (Y=3)) = P(Y=-3) + P(Y=3) = \frac{2}{8}$
$P(Z=1) = \frac{3}{8} + \frac{3}{8} = \frac{6}{8}$

Def: The expectation of a discrete random variable X is given by

$$E[X] = \sum_{\substack{k \text{ s.t.} \\ P(X=k) \neq 0}} k \cdot P(X=k)$$

Thm: If you have a r.v. $Y = g(X)$, then

$$E[Y] = E[g(X)] = \sum_{\substack{k \text{ s.t.} \\ P(X=k) \neq 0}} g(k) \cdot P(X=k)$$

① Bernoulli:
parameter: $p \in [0,1]$
pmf: $P(X=1) = p$, $P(X=0) = 1-p$
$E[X] = p$ $Var(X) = p(1-p)$

② Binomial

parameters: $p \in [0,1]$, $n \in \mathbb{N}$

pmf : $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0,1,2,\dots,n\}$

$E[X] = np$ $\quad Var(X) = np(1-p)$

③ Geometric

parameter : $p \in [0,1]$

pmf : $P(X=k) = (1-p)^{k-1} p$ for $k \in \mathbb{N}$

$E[X] = \frac{1}{p}$ $\quad Var(X) = \frac{1-p}{p^2}$

④ Poisson

parameter: $\lambda \in \mathbb{R}^+$

pmf : $P(X=k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda}$ for $k \in \mathbb{N}$

$E[X] = \lambda$ $\quad Var(X) = \lambda$

Def: A continuous random variable X has $P(X=k) = 0$ for all k.
But $P(X \in [a,b]) > 0$ for $[a,b]$ containing possible values of X.

Instead of the PMF we use the cumulative distribution function.

$F(k) = P(X \le k)$.

* This helps us calculate $P(X \in (a,b])$ because

$P(X \in [a,b]) = P(X \not< a, X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a)$

Def: We have another function associated with any continuous r.v. X called the probability density function.
It is defined to be the derivative of the CDF.

$f(k) = F'(k) = \frac{d}{dk} P(X \le k)$

Note: $P(X \le k) = \int_{-\infty}^{k} f(x)\,dx$ and $P(X \in [a,b]) = \int_a^b f(x)\,dx$

ex. Suppose X has a PDF of $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } 1 \le x \\ 0 & \text{otherwise} \end{cases}$

Let $Y = \frac{1}{X}$. Find the PDF of Y.

$F_Y(k) = P(Y \le k) = P(\frac{1}{X} \le k) = P(X \ge \frac{1}{k}) = \begin{cases} 1 & \text{if } k > 1 \\ \int_{\frac{1}{k}}^{\infty} \frac{1}{x^2}\,dx & \text{if } \frac{1}{k} \ge 1 \iff 1 \ge k \ge 0 \\ 0 & \text{if } k < 0 \end{cases}$

$f_Y(k) = \frac{d}{dk} F_Y(k) = \begin{cases} 1 & \text{if } 0 \le k \le 1 \\ 0 & \text{otherwise} \end{cases}$

$\int_{\frac{1}{k}}^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{\frac{1}{k}}^{\infty} = 0 - \left(-\frac{1}{\frac{1}{k}}\right) = k$

$\frac{d}{dk}(k) = 1$

$f_Y(k) = \frac{d}{dk} F_Y(k) = \begin{cases} 1 & \text{if } k \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

$\frac{d}{dk}(k) = 1$

So Y is actually a uniform($[0,1]$) r.v.

Def: One important use for PDF's is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

(f: PDF of X)

Thm: $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$

Def: For <u>both</u> discrete and continuous random variables, the <u>variance</u> is defined

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \underline{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

better to calculate, always use this

Def: The <u>standard deviation</u> is defined

$$SD(X) = \sqrt{Var(X)}$$

① Uniform

parameters: a, b s.t. $b > a$

CDF: $P(X \le k) = \frac{k-a}{b-a}$ if $k \in [a,b]$

PDF: $f(k) = \frac{1}{b-a}$ if $k \in [a,b]$

$\mathbb{E}[X] = \frac{a+b}{2}$ $Var(X) = \frac{(b-a)^2}{12}$

② Exponential

parameter: $\lambda \in \mathbb{R}^+$

CDF: $P(X \le k) = 1 - e^{-\lambda k}$ for $k \ge 0$

PDF: $f(k) = \lambda e^{-\lambda k}$ for $k \ge 0$

$\mathbb{E}[X] = \frac{1}{\lambda}$ $Var(X) = \frac{1}{\lambda^2}$

③ Normal (aka Gaussian)

parameters: μ, σ^2

PDF: $f(k) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$

$\mathbb{E}[X] = \mu$ $Var(X) = \sigma^2$

PDF: $f(k) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\mathbb{E}[X] = \mu \quad Var(X) = \sigma^2$

Def: It is possible to have multiple r.v.'s defined on the same probability space.
If X and Y are discrete, we define them using the joint probability mass function.

$$p(x,y) = \mathbb{P}(X=x, Y=y).$$

$$\mathbb{P}((X,Y) \in A) = \sum_{(x,y)\in A} p(x,y)$$

If X and Y are continuous, we define them using the joint probability density function.

$$\mathbb{P}((X,Y) \in R) = \iint_R f(x,y)\, dA$$

Note: If we are given the joint PMF of X and Y we can compute the individual PMF of X (which we call the marginal PMF of X) via

$$p_X(x) = \mathbb{P}(X=x) = \sum_{y \text{ s.t. } \mathbb{P}(X=x, Y=y)\neq 0} \mathbb{P}(X=x, Y=y) = \sum_{y \text{ s.t. } p(x,y)\neq 0} p(x,y)$$

Similarly, $p_Y(y) = \sum_{x \text{ s.t. } p(x,y)\neq 0} p(x,y)$

For continuous r.v.'s

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy \qquad f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$$

Def: We say X and Y are independent if

Discrete: $p(x,y) = p_X(x) \cdot p_Y(y)$ (i.e. $\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x) \cdot \mathbb{P}(Y=y)$) for all x, y.

Continuous: $f(x,y) = f_X(x) \cdot f_Y(y)$ for all x, y

Def: We can also define conditional probability mass functions by

$$p_X(x|Y=y) = \mathbb{P}(X=x|Y=y) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)} = \frac{p(x,y)}{p_Y(y)}$$

Def: We can define expectations of functions of joint random variables.

$$\mathbb{E}[g(X,Y)] = \sum_{\mathbb{P}(X=x,Y=y)\neq 0} g(x,y) \cdot \mathbb{P}(X=x, Y=y) \text{ if } X \text{ and } Y \text{ are discrete}$$

$$\mathbb{E}[g(X,Y)] = \iint g(x,y) f(x,y)\, dA \text{ if } X \text{ and } Y \text{ are continuous}$$

ex. Suppose X and Y have a joint pdf given by

$Y \backslash X$	0	1	2
2	$\frac{1}{10}$	$\frac{1}{5}$	0
3	$\frac{1}{5}$	$\frac{1}{4}$	0
4	0	0	$\frac{1}{4}$

Find $\mathbb{E}[X+XY]$

$= (0+0\cdot 2)\cdot\frac{1}{10} + (0+0\cdot 3)\cdot\frac{1}{5}$

$+ (1+1\cdot 2)\cdot\frac{1}{5} + (1+1\cdot 3)\cdot\frac{1}{4} + (2+2\cdot 4)\cdot\frac{1}{4}$

$= \frac{3}{5} + \frac{4}{4} + \frac{10}{4} = \frac{3}{5} + \frac{14}{4}$

Thm:
- For any X and Y, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- If X and Y are independent, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Note: We can in particular use the first part of this theorem to compute expectations of random variables that are "counting things", using the indicator trick.

ex. Randomly throw 20 balls into 5 boxes, Box A, Box B, Box C, Box D, and Box E.
Let N be the number of boxes with exactly 3 balls in them.
What is $\mathbb{E}[N]$?

Let $I_k = \begin{cases} 1 & \text{if box \#k has exactly 3 balls in it} \\ 0 & \text{otherwise} \end{cases}$

$N = I_1 + I_2 + I_3 + I_4 + I_5 \Rightarrow \mathbb{E}[N] = \mathbb{E}[I_1 + I_2 + I_3 + I_4 + I_5]$

$= \mathbb{E}[I_1] + \mathbb{E}[I_2] + \mathbb{E}[I_3] + \mathbb{E}[I_4] + \mathbb{E}[I_5]$.

$\mathbb{E}[I_1] = 1\cdot \mathbb{P}(\text{Box \#1 has exactly 3 balls}) = \mathbb{P}(\text{Binom}(20,\tfrac{1}{5}) = 3) = \binom{20}{3}\cdot(\tfrac{1}{5})^3\cdot(\tfrac{4}{5})^{17}$

$\mathbb{E}[I_2] = \binom{20}{3}\cdot(\tfrac{1}{5})^3\cdot(\tfrac{4}{5})^{17}$

$\mathbb{E}[N] = 5\cdot\frac{20!}{3!\,17!}(\tfrac{1}{5})^3(\tfrac{4}{5})^{17}$

Thm: Let X be a random variable with $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$.
Then, if $X_1, X_2, X_3, X_4, \ldots$ are an infinite sequence of random variables, all independent, and all with the same PMF (if discrete) or CDF (if continuous) as X,

$\mathbb{P}(N(n) < k)$

Then, if $X_1, X_2, \ldots$ and all with the same PMF (if discrete) or CDF (if continuous) as X,

then $\lim_{n\to\infty} P\left(\frac{X_1+X_2+\cdots+X_n-n\mu}{\sqrt{n\cdot\sigma^2}} \le k\right) = P(N(0,1)\le k)$

Less rigorous more useful version:

$$P\left(\frac{X_1+X_2+\cdots+X_n-n\mu}{\sqrt{n\cdot\sigma^2}} \le k\right) \approx P(N(0,1)\le k) \quad \text{if } n \text{ is "big"}$$

ex. Let $X_1, X_2, X_3, \ldots, X_{150}$ be independent uniform $[1,3]$ random variables.

Find $P(X_1+X_2+\cdots+X_{150} \le 270)$

$= P(X_1+X_2+\cdots+X_{150}-300 \le -30)$

$= P\left(\frac{X_1+X_2+\cdots+X_{150}-300}{10} \le -3\right)$

$\approx P(N(0,1)\le -8) = P(Z \le -3)$

$= \Phi(-3) \approx 0.00135$

$\mu = E[\text{uniform}[1,3]] = \frac{1+3}{2} = 2$

$\sigma^2 = \text{Var}[\text{uniform}[1,3]] = \frac{(3-1)^2}{12} = \frac{4}{12} = \frac{1}{3}$

$n\mu = 300$

$\sqrt{n\cdot\sigma^2} = \sqrt{300\cdot\frac{1}{3}} = \sqrt{\frac{300}{3}} = \sqrt{100} = 10$